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ON ACYCLICITY OF CIRCUITS OF A DIGRAPH AND THE DUAL CONCEPT

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I. INTRODUCTION

For a circuit C of a given digraph G, consider the reference direction. With respect to this reference direction, C (represented by the edge set) can be partitioned as $C=C^+\cup C^-$, $C^+\cap C^-=\phi$ where C^+ is the set of all the edges whose directions follow the reference. Define the acyclicity of C by

$$a(C)=\min\ (|C^+|,|C^-|).$$

The acyclicity of the whole graph G is

$$a(G)=\min\ (a(C))$$

where min ranges over all the circuits. Trivially, $a(C)\le\lfloor\frac{|C|}{2}\rfloor$.

Usually, C is called cyclic or acyclic corresponding to $a(C)=0$ or $a(C)>0$, respectively. G also is called acyclic if $a(G)>0$.

The acyclicities of circuits of a certain set cannot be independent. For example, consider three circuits C_1, C_2 and C_2' of G shown in Fig.1. It is easy to see that

$$a(C_1)+a(C_2)-a(C_2')\le 3.$$

Theorem 1 presents this kind of dependence relations.

The dependencies of circuits with respect to acyclicity lead to the concept of k-th acyclicity dominating set D of circuits, which is defined as a set such that

$$\min_{C\in D}\ a(C)\ge k \quad \text{implies} \quad a(G)\ge k.$$

Theorem 2 determins the minimum first acyclicity dominating set.

It follows the complete dual discussion which treats the co-circuits (cuts) and their co-acyclicity (strongness of connectivity).

All these results are a version of our earlier works[1,2].

2. DEPENDENCY OF CIRCUITS WITH RESPECT TO ACYCLICITY

In this paper, terms "circuit" and "cut" are used to denote the simple ones. Circuits C_1 and C_2 are said to be confluent if $C_1 \cap C_2$ forms a nonempty simple path. If C_1 and C_2 are confluent, $C_2' = C_1 \oplus C_2 = (C_1 \cup C_2) - (C_1 \cap C_2)$ is again a circuit.

Lemma 1: Suppose C_1, C_2 are confluent and let $C_2' = C_1 \oplus C_2$. Then

$$|C_2'| - a(C_2') \leq |C_1 \cup C_2| - (a(C_1) + a(C_2)).$$

Proof: Let p_i, q_i, r_i denote the numbers of edges whose directions are coincide or not with the reference direction of those circuits contained in $C_1 - C_2$, $C_1 \cap C_2$, $C_2 - C_1$, respectively, as indicated in Fig.1. Then,

$$\begin{aligned} a(C_1) + a(C_2) &= \min(p_1 + q_2,\ p_2 + q_1) + \min(r_1 + q_2,\ r_2 + q_1) \\ &\leq \min(p_1 + r_2 + q_1 + q_2,\ p_2 + r_1 + q_1 + q_2) \\ &= \min(p_1 + r_2,\ p_2 + r_1) + q_1 + q_2 \\ &= a(C_2') + |C_1 \cup C_2| - |C_1 \cup C_2|. \end{aligned}$$

Q.E.D.

Theorem 1: Let $(C_1, C_2, \ldots, C_n)$ be a sequence of circuits such that

$$C_k' = C_1 \oplus C_2 \ldots \oplus C_k$$

is a circuit and C_k' and C_{k+1} are confluent for $k = 1, 2, \ldots, n-1$. Then

$$|C_n'| - a(C_n) \leq |\bigcup_{i=1}^{n} C_i| - \sum_{i=1}^{n} a(C_i).$$

Proof: For $n=2$, the proposition is Lemma 2. Suppose it is true for $n \leq m-1$. By the lemma, since $C_m' = C_{m-1}' \oplus C_m$,

$$\begin{aligned} |C_m'| - a(C_m') &\leq |C_{m-1}' \cup C_m| - a(C_{m-1}') - a(C_m) \\ &= |C_{m-1}' \cup C_m| - |C_{m-1}'| + \{|C_{m-1}'| - a(C_{m-1}')\} - a(C_m) \\ &\leq |C_{m-1}' \cup C_m| - |C_{m-1}'| + \{|\overset{m-1}{\cup} C_i| - \overset{m-1}{\Sigma} a(C_i)\} - a(C_m) \\ &= |\overset{m}{\cup} C_i| - \overset{m}{\Sigma} a(C_i). \end{aligned}$$

Q.E.D.

A typical example is when $\cup C_i = E$ forms a planar subgraph G in which $C_1, C_2, \ldots, C_n$ are the inner meshes properly ordered and C_n' the outer mesh of G which is properly drawn on a plane. If we put $E_I = E - C_n'$, the set of inner edges, the theorem is

$$a(C_n') \geq \Sigma a(C_i) - |E_I|.$$

Thus, the theorem has a meaning only when $\Sigma a(C_i) - |E_I| > 0$.

III. THE k-TH ACYCLICITY DOMINATING SET

Let S be the set of all the circuits of G. A subset $D \subset S$ is called the k-th acyclicity dominating set if

(P) $a(C) \geq k$ (for all $C \in D$) implies $a(G) \geq k$.

A circuit C is called to associate the i-chord if the contraction of C produces a new circuit of length i. Let S(i) be the set of all circuits which are associated with i-chords.

Suppose C_1 and C_2 are confluent and $|C_1 \cap C_2| \leq k$. Then Lemma 1 insists that $a(C_1) \geq k$ and $a(C_2) \geq k$ lead to $a(C_1 \oplus C_2) \geq k$. Hence the lemma.

Lemma 2: For any $C \in \overset{k}{\cup} S(i)$, $S-\{C\}$ is a k-th acyclicity dominating set.

If G contains a circuit of length 2k-1 or less, it is trivial that $a(G) \not\geq k$. Hence the determination of the k-th acyclicity dominating set is meaningful only when G contains no circuit of length 2k-1 or less. That is, the girth of G is 2k or more.

Theorem 2: Suppose the girth of a digraph G is 2k+1 or more. Then, $D=S-\overset{k}{\cup} S(i)$ is a k-th acyclicity dominating set.

Proof: Consider a circuit $C \notin D$. There exist two other circuits C_1 and C_2 such that C_1 and C_2 are confluent, $C_1 \oplus C_2 = C$, and $|C_1 \cap C_2| \leq k$. Furthermore, both $|C_1|$ and $|C_2|$ are strictly less than $|C|$ because, for i,j=1,2

$$\begin{aligned} |C_i| &< |C_i| + (|C_j| - 2k) \\ &\leq |C_i| + |C_j| - 2|C_i \cap C_j| = |C|. \end{aligned}$$

If $C_i \notin D$, there is another pair (C_{i_1}, C_{i_2}) the length of each strictly less than that of C_i. Continuing the discussion, it is true that acyclicity being k or more or less of any circuit not in D can be checked by the members of D. Thus D is a k-th dominating set.

Q.E.D.

Most interesting is the consideration when k=1 for it distinct G between being acyclic or not. For this case, we can give the minimum dominating set.

Theorem 3: If the girth of G is 3 or more (i.e. G contains no parallel edges or self-loops), D=S-S(1) is the unique and minimum first acyclicity dominating set.

Proof: It suffices to show that for any $C \in D$ we cannot prove whether $a(C) \geq 1$ from the information of other circuits being acyclic.

Consider the graph G' which is obtained from G by contracting the edges of C to a vertex v. Suppose G is so oriented as: G' is acyclic and C cyclic.

Every graph has this possibility since G' contains no self-loops by assumption.

Then, in G, all circuits except C is acyclic. Hence S-{C} is not a dominating set.

Q.E.D.

IV. THE DUAL CONCEPT

We do not follow the dual discussion faithfully but, for reference, only the fact corresponding to Theorem 3 will be cited.

A 1-chorded cut is a cut which, after deletion of its edges, produce a bridge. A cut is co-acyclic if all of its edge do not follow the same direction.

Theorem 3': Suppose that G contains no cut of cardinality 2 or less. Then, the set D' of all cuts that are not 1-chorded is the minimum and unique set such that every cut of D' being co-acyclic implies G being strongly connected.

V. CONCLUSION

This paper studied the dependency of acyclisity of circuits. Theorem 3 and 3' can be extended to the general cases in which no restriction is imposed[1].

REFERENCES

[1] Y. Kajitani and F. Hirose,"On acycle basis and co-acycle basis," Proc. Tech. Group on Circuits and Systems, IECE of Japan, CST76-15, 1976.

[2] E. Okamoto, H. Satoh and Y. Kajitani,"The acyclicity of digraphs," Proc. 1977 Convention Record, 17, 1977.

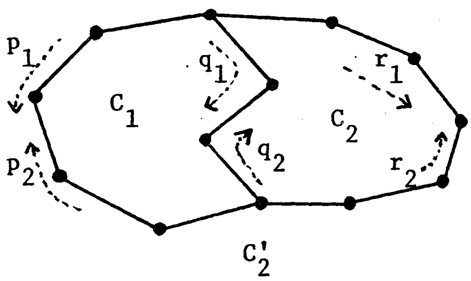

Fig.1 Confluent pair of circuits C_1 and C_2 and $C'_2=C_1 \oplus C_2$